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Pancyclic in-tournaments

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Abstract

An in-tournament is an oriented graph, where the negative neighborhood of every vertex induces a tournament. In this paper, the influence of the minimum indegree $\delta^-(D)$ of an in-tournament D on its k-pancyclicity is considered. An oriented graph of order n is said to be k-pancyclic for some $3 \leqslant k \leqslant n$, if it contains an oriented cycle of length t for every $k \leqslant t \leqslant n$. For every $3 \leqslant k \leqslant n$, a lower bound for $\delta^-(D)$ is presented that ensures a strong in-tournament to be k-pancyclic. Examples show that all bounds given here are best possible.

1. Terminology and introduction

All digraphs mentioned in this paper are *oriented graphs* which means that they contain no multiple arcs, no loops, and no cycles of length 2. An oriented graph that has no pair of non-adjacent vertices is called a *tournament*. If the positive as well as the negative neighborhood of every vertex of a directed graph D induces a tournament, then D is a *local tournament*. When this is only true for the negative neighborhood of every vertex, then D is called an *in-tournament*. Hence, every pair of distinct vertices of an in-tournament D that have a common positive neighbor in D is connected by exactly one arc.

A digraph D is determined by its set of vertices $V(D)$ and its set of arcs $E(D)$. If $xy \in E(D)$ for some distinct vertices $x, y \in V(D)$, then we say that *y is dominated by x* and *x dominates y*, denoted by $x \to y$. In other words, y is a *positive neighbor* of x and x is a *negative neighbor* of y. Let S be an arbitrary subset of $V(D)$ or a subdigraph of D. Then $D - S$ denotes the digraph that is induced by the vertices $V(D)\backslash S$ or $V(D)\backslash V(S)$, respectively. Let $x \in V(D)$ be an arbitrary vertex. The *negative neighborhood of x with respect to S*, $N^-(x,S)$, is the set of negative neighbors of x that belong to S. Analogously, $N^+(x,S)$ is the *positive neighborhood of x in S*. Define $d^-(x,S) = |N^-(x,S)|$ and $d^+(x,S) = |N^+(x,S)|$. If $S = V(D)$ or $S = D$, respectively, then we also write $N^-(x)$, $N^+(x)$, $d^-(x)$, and $d^+(x)$.

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If $d^-(x) = d^+(x) = r$ for every vertex $x \in V(D)$ for some integer $r \geqslant 1$, then the digraph D is called *r-regular*. The *minimum indegree* $\delta^-(D)$ and the *minimum out-degree* $\delta^+(D)$ of D is given by $\delta^-(D) = \min_{x \in V(D)} d^-(x)$ and $\delta^+(D) = \min_{x \in V(D)} d^+(x)$. Moreover, $\delta(D) = \min\{\delta^-(D),\ \delta^+(D)\}$ is the *minimum degree* of D. It is not difficult to see that $\delta^-(D), \delta^+(D) \leqslant (n-1)/2$, if D is an oriented graph.

Let S_1 and S_2 be two disjoint subsets of $V(D)$. The digraph that is induced by the vertices of S_1 is denoted by $D[S_1]$. If $s_1 \to s_2$ for every $s_1 \in S_1$ and $s_2 \in S_2$, then we write $S_1 \to S_2$. For $S_2 = \{s_2\}$, we use the short form $S_1 \to s_2$, instead of $S_1 \to \{s_2\}$. If, for example, S_1 is a subdigraph of D, we also use $S_1 \to S_2$ to denote the fact that $V(S_1) \to S_2$.

All *cycles* and *paths* mentioned here are oriented cycles and oriented paths. A cycle or a path that contains all the vertices of a digraph is a *Hamiltonian cycle* or a *Hamiltonian path*, respectively. Obviously, the smallest possible cycle in an oriented graph consists of three vertices. A cycle of length k is called a *k-cycle*. Let C be a non-Hamiltonian k-cycle in a digraph D. Then C is *extendable*, if D contains a $(k+1)$-cycle C^* such that $V(C) \subset V(C^*)$. A digraph D of order n is said to be *k-pancyclic* for some $3 \leqslant k \leqslant n$, if D has a t-cycle for every $k \leqslant t \leqslant n$. If even every vertex of D is contained in a cycle of length t for every $k \leqslant t \leqslant n$, then D is *vertex k-pancyclic*. For $k = 3$, we also use the terms *pancyclic* and *vertex pancyclic*.

Tournaments and their different generalizations form one of the most interesting fields in the investigation of digraphs. To obtain the class of digraphs that is considered in this paper, the general adjacency between every pair of distinct vertices in a tournament is transferred to a local property. In the first step, this leads us to local tournaments, where the positive as well as the negative neighborhood of every vertex induces a tournament. Since their introduction by Bang-Jensen [1] in 1990, the structure of these oriented graphs has been studied intensively. In particular, the Ph. D. theses of Guo [4] and Huang [6] have been devoted to the investigation of local tournaments and of the more general class of locally semicomplete digraphs, where the existence of 2-cycles is permitted.

Going further in this direction of generalization, in-tournaments are oriented graphs, where only the negative neighborhood of every vertex is asked to induce a tournament. In [3], Bang-Jensen et al. studied this larger class of digraphs. More work has been done in [9–11], where the authors focused on the cycle structure of in-tournaments. In the latter one, sufficient conditions for vertex k-pancyclic in-tournaments of order n based on the minimum degree were given for $k = 3, 4, 5$ and $k \geqslant n-3$, and it was conjectured that an analogue result holds for the remaining values of k.

In this paper, the influence of the minimum indegree $\delta^-(D)$ of an in-tournament D of order n on its k-pancyclicity is considered. This problem was already solved for $k = 3$ in [10], where a sharp lower bound for $\delta^-(D)$ was given that ensures a strong in-tournament to be pancyclic. Now, we present analogous results for every $3 \leqslant k \leqslant n$. All bounds for $\delta^-(D)$ given in this context are best possible. Moreover, the families of digraphs showing the sharpness of our results are local tournaments. Hence, all the statements are best possible even for this subclass of in-tournaments and provide a nice supplement to [2], where a structural characterization of k-pancyclic and vertex k-pancyclic local tournaments is given.

2. Preliminary results

In this section, two results on tournaments are listed. They will be applied to sub-tournaments induced by the negative neighborhood of certain vertices.

Theorem 2.1 (Rédei [8]). *Every tournament contains a Hamiltonian path.*

Theorem 2.2 (Harary, Moser [5]; Moon [7]). *Every strong tournament is pancyclic.*

The following theorem on mergable paths in in-tournaments is due to Bang-Jensen et al. [3]. In a special version, it will be useful in proving the main result of Section 3.

Theorem 2.3 (Bang-Jensen, Huang, Prisner [3]). *Let $P_1=x_1x_2\dots x_t$ and $P_2=y_1y_2\cdots y_s$ be two vertex disjoint paths in an in-tournament D. If there exist integers i and j, $1\leqslant i<j\leqslant t$, such that $x_i\to y_1$ and $y_s\to x_j$, then D contains a path P from x_1 to x_t with $V(P)=V(P_1)\cup V(P_2)$.*

3. In-tournaments without a cycle of length k

The first step in developing the above-mentioned conditions ensuring k-pancyclicity in in-tournaments for arbitrary k, is the consideration of in-tournaments D of order n with minimum indegree $\delta^-(D)>0$ that have no cycle of length k for some $3\leqslant k\leqslant n$. The obtained results will be the main tool for the subsequent investigation of k-pancyclicity.

To prove this statement, we use the following preparatory results.

Lemma 3.1. *Let $C=a_1a_2\dots a_ta_1$ and $P=b_1b_2\dots b_s$ be a cycle and a path of an in-tournament D such that $V(C)\cap V(P)=\emptyset$. If $P\to a_t$ and b_1 does not dominate C, then D contains a cycle C' with $V(C')=V(C)\cup V(P)$.*

Proof. Since b_1 does not dominate C, there exists an integer i, $1\leqslant i<t$, such that $a_i\to b_1$. To see this, note that b_1 and a_{t-1} are negative neighbors of a_t and therefore adjacent. If $a_{t-1}\to b_1$, then $i=t-1$. Otherwise, $b_1\to a_{t-1}$ implies that b_1 and a_{t-2} are adjacent and, since b_1 does not dominate C, we obtain i in at most $t-1$ steps.

Let $P_1=a_1a_2\dots a_t$ and $P_2=P$. Since $a_i\to b_1$ for some $1\leqslant i<t$ and $b_s\to a_t$, it follows by Theorem 2.3 that D contains a path P' from a_1 to a_t with $V(P')=V(P_1)\cup V(P_2)$. Then $C'=P'a_1$ is the desired cycle. $\square$

Lemma 3.2. *Let D be an in-tournament and let C be a non-Hamiltonian cycle in D such that $d^-(x^*, D-C) > 0$ for some $x^* \in V(C)$. If there exists a vertex $x \in V(C)$ such that $N^-(x) \subset V(C)$, then C is extendable.*

Proof. Let $C = u_1u_2 \dots u_tu_1$ and assume that $N^-(u_1) \subset V(C)$. By the hypothesis, there exists a vertex $z \in V(D-C)$ such that $z \to u_i$ for some $1 \leqslant i \leqslant t$. Let i be the smallest such integer. Clearly, $i \geqslant 2$. Now the vertices z and u_{i-1} are adjacent and the minimality of i implies that $u_{i-1} \to z$. Hence, C can be extended to the cycle $u_1 \dots u_{i-1}zu_i \dots u_tu_1$ and we are done. □

If it is possible to secure the existence of an arc in D leading from outside the cycle C to a vertex of C (for example, if D is strong), we can repeat the argumentation in the proof of Lemma 3.2 to obtain a sequence of extended cycles that terminates with a Hamiltonian cycle of D.

Corollary 3.3. *Let D be a strong in-tournament of order n containing a cycle C_0 such that $N^-(x) \subset V(C_0)$ for some $x \in V(C_0)$. Then there exists a sequence of cycles $C_0, C_1, \dots, C_t$, $t \geqslant 0$, in D such that $|V(C_t)| = n$, $|V(C_{i-1})| + 1 = |V(C_i)|$, and $V(C_{i-1}) \subset V(C_i)$ for every $1 \leqslant i \leqslant t$.*

Proof. Since D is strong, there is a vertex $x^* \in V(C_0)$ such that $d^-(x^*, D-C_0) > 0$. Lemma 3.2 implies the existence of the cycle C_1 with the desired properties. Now C_1 meets the hypothesis of Lemma 3.2 and successively we obtain the desired result. □

Now we come to the main result of this section in which we investigate in-tournaments without a cycle of length k.

Theorem 3.4. *Let D be an in-tournament of order n with minimum indegree $\delta^-(D) > 0$ that contains no cycle of length k for some $3 \leqslant k \leqslant n$. If $k \geqslant 2\delta^-(D) + 2$, let D be strong. Then there are $k+1$ disjoint non-empty vertex sets $N_0, N_1, \dots, N_k$ in $V(D)$ such that the following holds.*

1. *N_0 is the vertex set of a longest cycle on less than k vertices in D or, if such a cycle does not exist, N_0 consists of an arbitrary vertex of D.*
2. *$|N_i| \geqslant \delta^-(D) - (|N_0| - 1)/2$ for every $1 \leqslant i \leqslant k$.*
3. *If $|N_{i'}| = (2\delta^-(D) + 1)/3 - s$ for some $1 \leqslant i' \leqslant k$ and $s > 0$, then $|N_{i'-1}| \geqslant (2\delta^-(D) + 1)/3 + 2s$.*

Proof. Analogous to the description in the statement of the theorem, let C be a longest cycle on less than k vertices in D or, if such a cycle does not exist, let $V(C)$ consists of an arbitrary vertex of D. Let $N_0 = V(C)$, $|V(C)| = t$, and $p = \delta^-(D)$. Note that the definition of C implies that $1 \leqslant t \leqslant k-1$. In the main step of the proof we show that there are k disjoint non-empty vertex sets $N_1, N_2, \dots, N_k$ in $V(D-C)$ such that, for every $1 \leqslant i \leqslant k$, $D[N_i]$ is a tournament, $N_i \to x_{i-1}$ for some $x_{i-1} \in N_{i-1}$, and $|N_i| \geqslant p-(d_{i-1}-1)/2$ for some $d_{i-1} \leqslant \min\{t, |N_{i-1}|\}$. We proceed by induction on the number r of vertex sets, $1 \leqslant r \leqslant k$.

Consider the case, when $r=1$. Define $N_1 \subseteq V(D-C)$ such that $z \in N_1$ if and only if $d^+(z,C) > 0$, and let $n_1 = |N_1|$. Assume first that $t \geqslant 3$. The maximality of C yields that it is not extendable. For $P=z$ for an arbitrary vertex $z \in N_1$, it follows, therefore, by Lemma 3.1 that $z \to C$. We conclude that $N_1 \to C$ which implies that $D[N_1]$ is a tournament. Define $d_0 = t = |N_0|$. Since there exists a vertex $x_0 \in V(C)$ such that $d^-(x_0, C) \leqslant (d_0-1)/2$ and since $p \leqslant d^-(x_0) = d^-(x_0, C) + n_1 \leqslant (d_0-1)/2 + n_1$, it follows that $n_1 \geqslant p-(d_0-1)/2$ which proves the cardinality property for N_1. It remains to show that N_1 is non-empty. If $d_0 \leqslant 2p$, then $n_1 \geqslant p-(d_0-1)/2 > 0$. Otherwise, we have $k \geqslant t+1 = d_0+1 \geqslant 2p+2$ and the strong connectivity of D implies that N_1 consists of at least one vertex. If $t=d_0=1$, then obviously, $n_1 \geqslant p = p-(d_0-1)/2$. Moreover, N_1 is non-empty and $D[N_1]$ is a tournament.

Suppose now that the non-empty vertex sets $N_1, N_2, \ldots, N_r$ are defined for some $1 \leqslant r \leqslant k-1$ such that N_1 is given by the description above and for every $1 \leqslant i \leqslant r-1$, we have $N_{i+1} = N^-(x_i, D-(N_0 \cup \cdots \cup N_i))$ for some $x_i \in V(D_i) \subseteq N_i$, where D_i is the strong component of the tournament $D[N_i]$ such that there is no arc leading from $D[N_i]-D_i$ to D_i. Define $n_i = |N_i|$ and $d_i = |V(D_i)|$ for every $1 \leqslant i \leqslant r-1$. Obviously, $d_i \leqslant n_i$. Furthermore, the maximality of C together with Theorem 2.2 implies that $d_i \leqslant t$ and hence, $d_i \leqslant \min\{t, n_i\}$ for every $1 \leqslant i \leqslant r-1$. By the induction hypothesis, we have $n_i \geqslant p-(d_{i-1}-1)/2$ for every $1 \leqslant i \leqslant r$.

To define the vertex set N_{r+1} with the desired properties, consider analogously the negative neighborhood of the vertex $x_r \in V(D_r)$ with $d^-(x_r, D_r) \leqslant (|V(D_r)|-1)/2$, where D_r is the strong component of the tournament $D[N_r]$ such that there is no arc leading from $D[N_r]-D_r$ to D_r. This is possible, since N_r is non-empty. Analogously, let $n_r = |N_r|$ and $d_r = |V(D_r)|$, where $d_r \leqslant n_r$ is obvious and $d_r \leqslant t$ by Theorem 2.2. Now Theorem 2.1 implies that there exists a Hamiltonian path $b_1 b_2 \ldots b_{n_r}$ of $D[N_r]$ such that, without loss of generality, $b_1 = x_r$. To examine the negative neighborhood of the vertex x_r in $\bigcup_{i=0}^{r-1} N_i$, assume that $y \to x_r$ for some $y \in \bigcup_{i=0}^{r-1} N_i$.

Let $y \in V(D_{r-1})$ first. In particular, $x_r = b_1$ does not dominate $V(D_{r-1})$ in this case. By its definition, $N_r \to x_{r-1}$ and it follows by Lemma 3.1 that for every $1 \leqslant i \leqslant n_r$, the vertex set of the path $b_1 b_2 \ldots b_i$ together with the Hamiltonian cycle of D_{r-1} form a cycle C_i of length $d_{r-1}+i$. Since $d_{r-1} \leqslant t \leqslant k-1$ and D contains no cycle of length k, we conclude that $d_{r-1}+n_r \leqslant k-1$. Consider the $(d_{r-1}+n_r)$-cycle C_{n_r}. Obviously, C_{n_r} contains x_{r-1} and all its negative neighbors. For $|V(C_{n_r})| \leqslant 2p$, it is easy to see that there exists a vertex $x^* \in V(C_{n_r})$ such that $d^-(x^*, D-C_{n_r}) > 0$. If $|V(C_{n_r})| = d_{r-1}+n_r \geqslant 2p+1$, then $k \geqslant d_{r-1}+n_r+1 \geqslant 2p+2$ and hence, the strong connectivity of D implies the existence of such a vertex $x^* \in V(C_{n_r})$. By Lemma 3.2, C_{n_r} is extendable. By the same arguments, we can show the necessary properties for the resulting cycle and hence, Lemma 3.2 yields the existence of a k-cycle in D, a contradiction. Therefore, $x_r \to D_{r-1}$ which implies that $d^-(x_r, D_{r-1}) = 0$. Analogously, it is possible to verify that $x_{i+1} \to D_i$ for every $1 \leqslant i \leqslant r-2$.

Assume now that $y \in N_{r-1} \backslash V(D_{r-1})$. If $d_{r-1} \geqslant 3$, let $a_1 a_2 \ldots a_{d_{r-1}} a_1$ be a Hamiltonian cycle of D_{r-1} such that $a_1 = x_{r-1}$. Otherwise, let $a_1 = x_{r-1}$. Since $D_{r-1} \rightarrow N_{r-1} \backslash V(D_{r-1})$, we conclude that D contains the cycle $y b_1 b_2 \ldots b_i a_1 a_2 \ldots a_j y$ of length $i + j + 1$ for every $1 \leqslant i \leqslant n_r$ and $1 \leqslant j \leqslant d_{r-1}$. Again, we have $k - 1 \geqslant d_{r-1} + n_r + 1$ and analogously to the case above, when $y \in V(D_{r-1})$, the application of Lemma 3.2 to the $(d_{r-1} + n_r + 1)$-cycle $y b_1 b_2 \ldots b_{n_r} a_1 a_2 \ldots a_{d_{r-1}} y$, which contains the vertex a_1 and $N^-(a_1)$, leads to a contradiction.

Finally, let $y \in N_j$ for some $0 \leqslant j \leqslant r-2$. Since $x_1 \rightarrow C$ and $N_{i+1} \rightarrow x_i \rightarrow N_i \backslash V(D_i)$ for every $1 \leqslant i \leqslant r-1$, we obtain the cycle $y x_r x_{r-1} \ldots x_1 y$ of length $r+1$ or the $(r-j+2)$-cycle $y x_r x_{r-1} x_{r-2} \ldots x_j y$ in D, if $y \in N_0$ or $y \in N_j \backslash V(D_j)$ for some $1 \leqslant j \leqslant r-2$. Since $r \leqslant k-1$, it follows that $r+1 \leqslant k$ and $r-j+2 \leqslant k-1-1+2=k$. Again, we can include arbitrarily many vertices of N_r and $V(D_{r-1})$ to either of these cycles to obtain a cycle of length k or a cycle that can be extended by Lemma 3.2, a contradiction. Now let $y \in V(D_j)$ for some $1 \leqslant j \leqslant r-2$. Obviously, D contains the cycle $y x_r x_{r-1} \ldots x_{j+1} y$ of length $r-j+1$, where $r-j+1 \leqslant k-1$. Define P to be a path from x_j to y in D_j which obviously exists, since D_j is a strong component and $y, x_j \in V(D_j)$. Clearly, $1 \leqslant |V(P)| \leqslant d_j$. Since $x_{j+1} \rightarrow D_j$, we can successively add all the vertices of P to the mentioned $(r-j+1)$-cycle to obtain the cycle $y x_r x_{r-1} \ldots x_{j+1} P y$. By the assumption, this cycle contains less than k vertices. Now we can add arbitrarily many vertices of N_r and D_{r-1} which leads to the above contradiction. Note that x_j is the initial vertex of P and so it is possible to include vertices of D_{r-1} in the cycle in consideration even if $j = r-2$.

We summarize that $d^-(x_r, N_0 \cup \cdots \cup N_r) = d^-(x_r, N_r) = d^-(x_r, D_r) \leqslant (d_r - 1)/2$. Hence, define $N_{r+1} = N^-(x_r, D - (N_0 \cup \cdots \cup N_r))$ which implies that $|N_{r+1}| \geqslant p - (d_r - 1)/2$ and $D[N_{r+1}]$ is a tournament. It remains to show that N_{r+1} is non-empty. For $d_r \leqslant 2p$, this is obvious. If $d_r \geqslant 2p+1 \geqslant 3$, we have $k \geqslant t+1 \geqslant d_r + 1 \geqslant 2p+2$ and the hypothesis implies that D is strong. If $N_{r+1} = \emptyset$, then the definition of N_{r+1} yields that $N^-(x_r) \subset V(D_r)$, and we derive a contradiction by applying Corollary 3.3 to the Hamiltonian cycle of D_r.

Altogether, we have proved that there are $k+1$ disjoint non-empty vertex sets $N_0, N_1, \ldots, N_k$ in $V(D)$ such that $n_0 = |N_0| = t$ and $n_i = |N_i| \geqslant p - (d_{i-1} - 1)/2$ for every $1 \leqslant i \leqslant k$, where $d_{i-1} \leqslant \min\{n_0, n_{i-1}\}$. This implies immediately that $n_i \geqslant p - (n_0 - 1)/2$ for every $1 \leqslant i \leqslant k$ and hence, the second part of the theorem.

To show the final statement, assume that $n_{i'} = (2p+1)/3 - s$ for some $1 \leqslant i' \leqslant k$ and some $s > 0$. Since $n_{i'} \geqslant p - (d_{i'-1} - 1)/2$, this implies that $p - (d_{i'-1} - 1)/2 \leqslant (2p+1)/3 - s$. It is easy to see that this leads to $d_{i'-1} \geqslant (2p+1)/3 + 2s$, and since $d_{i'-1} \leqslant n_{i'-1}$, we obtain the desired inequality. □

The sequence of disjoint vertex sets $N_0, N_1, \ldots, N_k$ that is described in Theorem 3.4 is called a *neighborhood sequence of order* $k+1$ *with initial point* C, where C denotes the Hamiltonian cycle of $D[N_0]$.

In Theorem 3.4, the condition concerning the strong connectivity of D in the case, when $k \geqslant 2\delta^-(D)+2$ cannot be weakened. To see this, consider the digraph D of order

$n = 2p + 1 + b$, where $p \geqslant 1$ and $1 \leqslant b \leqslant 2p + 1$ are two integers. Let $V(D) = A \cup B$ such that $A \cap B = \emptyset$, $|A| = 2p + 1$, $|B| = b$, and $A \to B$. Let $D[A]$ be a p-regular tournament and let B induce a transitive tournament. Obviously, D is a non-strong in-tournament with $\delta^-(D) = p$. Moreover, D contains no cycle of length k for every $k \geqslant 2p + 2 = 2\delta^-(D) + 2$. If D would have a neighborhood sequence $N_0, N_1, \ldots, N_k$ for some $k \geqslant 2p + 2$, then the Hamiltonian cycle C of $D[A]$ would be its initial point. But clearly, there are no $k \geqslant 2p + 2$ disjoint non-empty vertex sets $N_1, N_2, \ldots, N_k$ in $D - C = D[B]$.

4. On the existence of *k*-cycles

We present two different bounds for the minimum indegree $\delta^-(D)$ of an in-tournament D of order n that ensure the existence of a cycle of length k in D. Each of these bounds is valid for every $3 \leqslant k \leqslant n$, but it turns out that the first one is best possible for large values of k, which means $k \geqslant \sqrt{n+1}$, while for the remaining values of k, the second one is sharp.

Theorem 4.1. *Let D be an in-tournament of order n, and let $3 \leqslant k \leqslant n$ be an integer such that $\delta^-(D) > 3n/(2k+2) - \frac{1}{2}$. Furthermore, let D be strong if $k \geqslant 2\delta^-(D) + 2$. Then D contains a cycle of length k.*

Proof. Let $p = \delta^-(D)$ and suppose to the contrary that D contains no cycle of length k. By Theorem 3.4, D has a neighborhood sequence $N_0, N_1, \ldots, N_k$ of order $k + 1$. For $t = |N_0|$, the second part of Theorem 3.4 yields

$$n \geqslant \sum_{i=0}^{k} |N_i| \geqslant t + k\left(p - \frac{t-1}{2}\right) = k\left(p + \frac{1}{2}\right) - t\left(\frac{k}{2} - 1\right).$$

If $t \leqslant n/(k+1)$, then this implies

$$n \geqslant k\left(p + \frac{1}{2}\right) - \frac{n}{k+1}\left(\frac{k}{2} - 1\right)$$

and we derive the contradiction $p \leqslant 3n/(2k+2) - \frac{1}{2}$.

For $t > n/(k+1)$, define i_0, $0 \leqslant i_0 \leqslant k$, to be the maximal integer such that $|N_{i_0}| < n/(k+1)$. Since we consider $k+1$ vertex sets and we have $|N_0| = t > n/(k+1)$, such an index exists and it follows that $i_0 \geqslant 1$. The maximality of i_0 implies that $\sum_{i=i_0+1}^{k} |N_i| \geqslant (k - i_0)n/(k+1)$. It follows from the hypothesis, that $n/(k+1) < (2p+1)/3$ and hence, $|N_{i_0}| = (2p + 1)/3 - s$ for some $s > 0$. Now the third part of Theorem 3.4 implies that $|N_{i_0-1}| \geqslant (2p+1)/3 + 2s$ and therefore, $|N_{i_0}| + |N_{i_0-1}| \geqslant 2(2p+1)/3 + s > 2n/(k+1)$. This yields $\sum_{i=i_0-1}^{k} |N_i| > (k - i_0 + 2)n/(k+1)$. For $i_0 > 1$, let analogously i_1, $1 \leqslant i_1 \leqslant i_0 - 1$, be the maximal integer such that $|N_{i_1}| < n/(k+1)$. If i_1 does not exist, we summarize that $\sum_{i=0}^{k} |N_i| > (k+1)n/(k+1) = n$, a contradiction. Otherwise, $|N_{i_0-1}| \geqslant (2p + 1)/3 + 2s > n/(k+1)$ implies that $i_1 \leqslant i_0 - 2$, and we proceed with the above argumentation to derive the contradiction that $\sum_{i=0}^{k} |N_i| > (k+1)n/(k+1)$ in at most $(i_0-1)/2$ steps. □

The following family of examples shows that the bound $3n/(2k+2) - \frac{1}{2}$ for the minimum indegree in Theorem 4.1 is best possible, if $k \geqslant \sqrt{n+1}$. For integers $k \geqslant 3$ and $1 \leqslant m \leqslant k/2$, let D_m^k be the digraph of order $n = (2m-1)(k+1)$ with the vertex set $V(D_m^k) = V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$, and $|V_i| = 2m-1$ for every $1 \leqslant i \leqslant k+1$. Moreover, let V_i induce an $(m-1)$-regular tournament and let $V_i \to V_{i+1}$ for every $1 \leqslant i \leqslant k+1$, where $k+2=1$. It is easy to see that D_m^k is a strong in-tournament with $\delta^-(D_m^k) = m-1+2m-1 = 3m-2 = 3n/(2k+2) - \frac{1}{2}$. Since $m \leqslant k/2$, we have $|V_i| = 2m-1 \leqslant k-1$ and hence, D contains no cycle of length k.

On the other hand, $n = (2m-1)(k+1) \leqslant (k-1)(k+1) = k^2-1$. In varying the cardinality of the vertex sets V_i, the above construction yields the desired examples for arbitrary $k \geqslant \sqrt{n+1}$.

Note that the members of the family D_m^k are not only in-tournaments but even local tournaments with $\delta(D_m^k) = \delta^-(D_m^k) = 3n/(2k+2) - \frac{1}{2}$. Hence, the lower bound $3n/(2k+2) - \frac{1}{2}$ cannot be weakened if we restrict ourselves to the smaller class of local tournaments or if the minimum degree instead of the minimum indegree is considered.

Now we turn our attention to the second condition on $\delta^-(D)$ that will be relevant for small values of k.

Definition 4.2. Let $n \geqslant 3$ be an integer. For integers $3 \leqslant k \leqslant n$, define the function

$$f(k) = \begin{cases} \dfrac{n+1}{k} + \dfrac{k-4}{2} & \text{for even } k, \\[2ex] \dfrac{n+2}{k} + \dfrac{k-5}{2} & \text{for odd } k. \end{cases}$$

Theorem 4.3. *Let D be an in-tournament of order n, and let $3 \leqslant k \leqslant n$ be an integer such that $\delta^-(D) > f(k)$. Furthermore, let D be strong if $k \geqslant 2\delta^-(D)+2$. Then D contains a cycle of length k.*

Proof. Let $\delta^-(D) = p$, and suppose to the contrary that D has no cycle of length k. Again, Theorem 3.4 implies that D has a neighborhood sequence $N_0, N_1, \ldots, N_k$ of order $k+1$, and by its second part, we obtain analogously to the proof of Theorem 4.1 that

$$n \geqslant \sum_{i=0}^{k} |N_i| \geqslant t + k\left(p - \frac{t-1}{2}\right) = k\left(p + \frac{1}{2}\right) - t\left(\frac{k}{2} - 1\right),$$

where $t = |N_0|$.

Consider the case, when k is even first. Since $t \leqslant k-1$, the above yields

$$n \geqslant k\left(p + \frac{1}{2}\right) - (k-1)\left(\frac{k}{2} - 1\right) = k(p+2) - k\frac{k}{2} - 1,$$

and we derive the contradiction $p \leqslant (n+1)/k + (k-4)/2 = f(k)$.

If k is odd, let $t \leqslant k-2$ first. Analogously, we deduce that

$$n \geqslant k\left(p+\frac{1}{2}\right)-(k-2)\left(\frac{k}{2}-1\right)=k\left(p+\frac{5}{2}\right)-k\frac{k}{2}-2$$

which implies the contradiction $p \leqslant (n+2)/k+(k-5)/2=f(k)$.

For $t=k-1$, we see that $t-1=k-2$ is not divisible by 2. Hence, the lower bound $p-(t-1)/2$ for the cardinality of the vertex sets N_i, $1 \leqslant i \leqslant k$, can be sharpened to $p-(t-2)/2$. It follows that

$$n \geqslant t+k\left(p-\frac{t-2}{2}\right)=k(p+1)-t\left(\frac{k}{2}-1\right)=k(p+1)-(k-1)\left(\frac{k}{2}-1\right)$$

$$=k\left(p+\frac{5}{2}\right)-k\frac{k}{2}-1$$

and we are done by the contradiction $p \leqslant (n+1)/k+(k-5)/2<f(k)$. □

To see that the bound $f(k)$ for the minimum indegree in Theorem 4.3 is best possible for the small values of k that are not covered by the examples following Theorem 4.1, let $k \geqslant 4$ be an even integer first. Consider the family of digraphs E_m^k of order $n=(k-1)(mk+1)$, where $m \geqslant 1$ is an arbitrary integer. Let $V(E_m^k)=V_1 \cup V_2 \cup \cdots \cup V_{mk+1}$ such that $V_i \cap V_j=\emptyset$ for $i \neq j$. Let $|V_i|=k-1$ and let V_i induce a $((k-2)/2)$-regular tournament for every $1 \leqslant i \leqslant mk+1$. Furthermore, for $1 \leqslant i \leqslant mk+1$, let $V_j \rightarrow V_i$ for every $i-m \leqslant j \leqslant i-1$ (all indices are taken modulo $mk+1$). Clearly, E_m^k is a strong in-tournament with $\delta^-(E_m^k)=(k-2)/2+m(k-1)=(n+1)/k+(k-4)/2$ and E_m^k contains no cycle of length k.

For odd $k \geqslant 3$, let analogously O_m^k be the family of digraphs of order $n=(k-2)(mk+1)$ for some integer $m \geqslant 1$ with the vertex set $V(O_m^k)=V_1 \cup V_2 \cup \cdots \cup V_{mk+1}$ such that $V_i \cap V_j=\emptyset$ for $i \neq j$. In this case, let $|V_i|=k-2$ and let $D[V_i]$ be a $((k-3)/2)$-regular tournament for every $1 \leqslant i \leqslant mk+1$. Again, let $V_j \rightarrow V_i$ for every $i-m \leqslant j \leqslant i-1$ for each $1 \leqslant i \leqslant mk+1$. Then O_m^k is a strong in-tournament with $\delta^-(O_m^k)=(k-3)/2+m(k-2)=(n+2)/k+(k-5)/2$ and O_m^k has no k-cycle.

Note that $m \geqslant 1$ yields $n \geqslant (k-1)(k+1)$ and $n \geqslant (k-2)(k+1)$, respectively. In enlarging the number $mk+1$ of vertex sets V_i, we obtain examples for arbitrary small $k \leqslant \sqrt{n+1}$.

Again, we underline that the families E_m^k and O_m^k consist of strong local tournaments with minimum degree $(n+1)/k+(k-4)/2$ and $(n+2)/k+(k-5)/2$, respectively. Therefore, these bounds are even best possible if they are related to the minimum degree or to local tournaments.

The example that illustrates the necessity of considering strong in-tournaments D in Theorem 3.4, if $k \geqslant 2\delta^-(D)+2$, shows that this condition cannot be dropped in Theorems 4.1 and 4.3 either. In varying the cardinality of the vertex set B one can construct non-strong in-tournaments with minimum indegree $\delta^-(D)$ for every $1 \leqslant \delta^-(D) \leqslant (n-2)/2$ that contain no cycle of length $2\delta^-(D)+2$.

5. k-pancyclic in-tournaments

As it was mentioned in the preceding section, the condition $\delta^-(D) > 3n/(2k+2) - \frac{1}{2}$ is best possible to verify the existence of a cycle of length k in an in-tournament D of order n, if k is at least $\sqrt{n+1}$. On the other hand, the bound $\delta^-(D) > f(k)$ is sharp for the remaining values of k. Now we show that both conditions imply not only the existence of a k-cycle in D, but even the k-pancyclicity of D. Since we are dealing with Hamiltonian cycles now, we obviously need to restrict ourselves to strong in-tournaments.

Clearly, $3n/(2k+2) - \frac{1}{2}$ is a decreasing function in k. Hence, Theorem 4.1 immediately implies the following result.

Corollary 5.1. *Let D be a strong in-tournament of order n such that $\delta^-(D) > 3n/(2k+2) - \frac{1}{2}$ for some integer $3 \leqslant k \leqslant n$. Then D is k-pancyclic.*

It follows from the observations in Section 4 that the above bound for $\delta^-(D)$ is best possible for $k \geqslant \sqrt{n+1}$.

For $k \leqslant \sqrt{n+1}$, we consider the bound $f(k)$ that was presented in Definition 4.2. Firstly, investigate the function f.

Lemma 5.2. *Let $n \geqslant 3$ and $3 \leqslant k \leqslant \sqrt{n+1}$ be two integers. Then $f(k) \geqslant f(k+1)$ and $f(k) \geqslant f(k+2)$.*

Proof. If k is even, then

$$f(k) - f(k+1) = \frac{n+1}{k} + \frac{k-4}{2} - \frac{n+2}{k+1} - \frac{k-4}{2}$$
$$= \frac{(k+1)(n+1) - k(n+2)}{k(k+1)} = \frac{n+1-k}{k(k+1)} > 0.$$

Moreover,

$$f(k) - f(k+2) = \frac{n+1}{k} + \frac{k-4}{2} - \frac{n+1}{k+2} - \frac{k-2}{2}$$
$$= \frac{(k+2)(n+1) - k(n+k+3)}{k(k+2)} = \frac{2n+2-k^2-2k}{k(k+2)}$$

which is non-negative for $k \leqslant \sqrt{n+1}$.

For odd k, we obtain analogously that

$$f(k) - f(k+1) = \frac{n+2}{k} + \frac{k-5}{2} - \frac{n+1}{k+1} - \frac{k-3}{2}$$
$$= \frac{(k+1)(n+2) - k(n+2+k)}{k(k+1)} = \frac{n+2-k^2}{k(k+1)}$$

and

$$f(k)-f(k+2)=\frac{n+2}{k}+\frac{k-5}{2}-\frac{n+2}{k+2}-\frac{k-3}{2},$$

where both terms are non-negative for $k\leqslant\sqrt{n+1}$. $\square$

Theorem 5.3. *Let D be a strong in-tournament of order n such that $\delta^-(D)>f(k)$ for some integer $3\leqslant k\leqslant\sqrt{n+1}$. Then D is k-pancyclic.*

Proof. Let $p=\delta^-(D)$ and $\alpha=\lfloor\sqrt{n+1}\rfloor$. By the hypothesis, $p>f(k)$, where $k\leqslant\alpha$. Since $f(t)\geqslant f(t+1)$ for every $k\leqslant t\leqslant\alpha$ and $f(\alpha)\geqslant f(\alpha+2)$ by Lemma 5.2, we have $p>f(t)$ for every integer t with $k\leqslant t\leqslant\alpha+2$. Therefore, Theorem 4.3 implies that D contains a t-cycle for each of these t.

For cycle lengths larger than $\alpha+2$, consider the value of $f(\alpha)$. Note that the above implies in particular that $p>f(\alpha)$. We show that $f(\alpha)\geqslant g(\alpha+3)$, where $g(k)=3n/(2k+2)-\frac{1}{2}$. Then $p>f(\alpha)\geqslant g(\alpha+3)\geqslant g(t)$ for every $\alpha+3\leqslant t\leqslant n$, since $g(k)$ is a decreasing function in k. By Theorem 4.1, D contains a cycle of length t for $\alpha+3\leqslant t\leqslant n$, which completes the proof.

Since $\sqrt{n+1}\geqslant\alpha\geqslant\sqrt{n+1}-1$, we obtain for even α

$$f(\alpha)=\frac{n+1}{\alpha}+\frac{\alpha-4}{2}\geqslant\frac{n+1}{\sqrt{n+1}}+\frac{\sqrt{n+1}-5}{2}=\frac{3}{2}\sqrt{n+1}-\frac{5}{2}.$$

For odd α, we deduce analogously that

$$f(\alpha)=\frac{n+2}{\alpha}+\frac{\alpha-5}{2}\geqslant\frac{3}{2}\sqrt{n+1}-3+\frac{1}{\sqrt{n+1}},$$

which altogether leads to $f(\alpha)\geqslant\frac{3}{2}\sqrt{n+1}-3+1/(\sqrt{n+1})$.

On the other hand, consider $g(\alpha+3)$. We have

$$g(\alpha+3)=\frac{3n}{2(\alpha+4)}-\frac{1}{2}\leqslant\frac{3}{2}\left(\frac{n}{\sqrt{n+1}+3}-\frac{1}{3}\right).$$

Since $n=(\sqrt{n+1}+3)^2-6(\sqrt{n+1}+3)+8$, this leads to

$$\begin{aligned}g(\alpha+3)&\leqslant\frac{3}{2}\left(\sqrt{n+1}+3-6+\frac{8}{\sqrt{n+1}+3}-\frac{1}{3}\right)\\&=\frac{3}{2}\sqrt{n+1}-5+\frac{12}{\sqrt{n+1}+3}.\end{aligned}$$

It is not difficult to check that $\frac{3}{2}\sqrt{n+1}-5+12/(\sqrt{n+1}+3)\leqslant\frac{3}{2}\sqrt{n+1}-3+1/(\sqrt{n+1})$ for $n\geqslant 3$. Hence, $f(\alpha)\geqslant g(\alpha+3)$ and we are done. $\square$

Again, the corresponding family of examples given in Section 4 show the sharpness of Theorem 5.3.

Like it was mentioned in the introduction, the special case of 3-pancyclic in-tournaments was already considered in [10]. For $k=3$, Theorem 5.3 states that every strong in-tournament of order n with $\delta^-(D)>(n-1)/3$ is pancyclic. This is equivalent to Corollary 4.6 in [10].

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